β of the set is different from zero. It follows, therefore, that they are satisfied only by

$$C_{k+1,p} = C_{k+2,p} = \cdots = C_{n,p} = 0;$$

and this result holds when $p = 1, 2, \cdots, k$. If now we substitute these zeros in the first k equations (1), we obtain a set of k equations of the type considered in Case II. We see therefore that it is possible to find values of C_{ij} $[i = 1, 2, \cdots, n; j = 1, 2, \cdots, k]$ which satisfy the first k equations of (1) together with those of equations (2) which do not involve the subscripts $k + 1, k + 2, \cdots, n$. The last l equations (1) can not be satisfied in conjunction with those of equations (2) which involve these subscripts.

From the point of view of economics this result means that there are s separate sets of solvent markets (corresponding to the s simple matrices of type (**a**)) such that no two markets of different sets have any dealings with each other. The rate of exchange between two such markets is indeterminate, but between two markets of the same set it is determined uniquely as in Case II. The l markets not included in these s sets are bankrupt; whereas the credit of each of them at any solvent market is zero, some of them have debit balances with at least one of the solvent markets.

A SIMPLE THEORY OF COMPETITION.

By GRIFFITH C. EVANS, The Rice Institute, Houston, Texas.

1. Postulates. Significant theories of economics may be discussed mathematically in very simple terms, and results may be stated, in precise fashion, which a non-mathematical theory could not be expected to deduce at all. Moreover the assumptions are simple enough to be verified approximately in actual societies, and yield results which may be easily calculated in definite cases. Such a rudimentary theory of competition is given below, in terms of linear and quadratic functions, and the elements of the calculus.

Let

$$q(u) = Au^2 + Bu + C \tag{1}$$

represent, as a quadratic function, the total cost of producing (that is, putting on the market) u units of a commodity in unit time. Let

$$y = ap + b \tag{2}$$

represent, as a linear function, the amount of goods, y, which, if the price is p, will be bought in the market in unit time.

Suppose there are two producers, each manufacturing, subject to the same cost function given by (1), amounts u_1 and u_2, respectively, in unit time, and each trying to make his profit a maximum. What will be the amounts u_1 and u_2 manufactured, and the price p at which the goods are sold? This is the problem of competition expressed in simplest terms.

If we denote the respective profits by π_1 and π_2, we shall have, in a situation where as much is sold as is produced, the equations

$$\pi_i = pu_i - q(u_i), \qquad i = 1, 2, \tag{3}$$
$$y = u_1 + u_2 = ap + b, \tag{4}$$

since pu_i is the selling value of the amount u_i. It remains however to add a determining postulate for the stationa y values of the quantities π_i.

1.1. As a possible determining postulate we may take that of Cournot:[1]

(*a*) *Each competitor assumes that the production of the other or others is independent of his, and tries to make his profit a maximum.*

The mathematical expression of this postulate is to regard u_1 and u_2 as independent variables, and write

$$\frac{\partial \pi_1}{\partial u_1} = 0, \qquad \frac{\partial \pi_2}{\partial u_2} = 0, \tag{5}$$

that is,

$$0 = p + u_i \frac{\partial p}{\partial u_i} - q'(u_i), \qquad i = 1, 2,$$

and since, by (4), $\partial p/\partial u_i = 1/a$,

$$0 = p + \frac{u_i}{a} - 2Au_i - B, \qquad i = 1, 2. \tag{5'}$$

If from these equations we eliminate u_1 and u_2 by adding them together and then making use of (4), we shall have a single equation which we may solve immediately for p and obtain

$$p = \frac{b - 2Aab - 2Ba}{- a(3 - 2Aa)}, \qquad u_1 = u_2 = \frac{b + Ba}{3 - 2Aa}. \tag{6}$$

1.2. It may be noticed that the postulate just used is not entirely equivalent to this other:

(*c*) *Each producer tries to determine the amount u_i of his production per unit time so as to make the total profit a maximum.*

The total profit is given by

$$\pi = (u_1 + u_2)p - q(u_1) - q(u_2),$$

and the condition which expresses this postulate is

$$0 = \frac{\partial \pi}{\partial u_1} = \frac{\partial \pi}{\partial u_2}, \tag{7}$$

or

$$\begin{aligned} 0 &= p + (u_1 + u_2)\frac{\partial p}{\partial u_i} - 2Au_i - B \\ &= p + \frac{ap + b}{a} - 2Au_i - B, \end{aligned}$$

[1] Augustin Cournot, *Recherches sur les principes mathématiques de la théorie des richesses*, Paris, 1838; translated by N. T. Bacon, *Researches into the Mathematical Principles of the Theory of Wealth*, London, 1897; see chapter VII.

so that, in the same way as in the previous case, the u_1 and u_2 may be eliminated by means of (4) and the value of p obtained:

$$p = \frac{2b - 2Aab - 2Ba}{-a(4 - 2Aa)}, \qquad u_1 = u_2 = \frac{b + Ba}{4 - 2Aa}. \tag{8}$$

This latter phenomenon we may define as *coöperation.*

1.3. An instructive standard of comparison is the monopoly price. For this, the natural postulate was given by Cournot:[1]

(*m*) *The producer determines his production per unit time so as to make his profit a maximum.*

In this case

$$\pi = pu - q(u),$$

and the condition is simply

$$\frac{d\pi}{du} = 0,$$

that is,

$$0 = p + \frac{u}{a} - 2Au - B.$$

Hence

$$p = \frac{b - 2Aab - Ba}{-a(2 - 2Aa)}, \qquad u = \frac{b + Ba}{2 - 2Aa}. \tag{9}$$

1.4. There is another situation which it is worth while to describe; and it is a second kind of competition. Each competitor may by slightly changing the price—say by underselling the other—try to obtain whatever portion of the trade he can handle with the maximum return or profit. That is, he regards the price as fixed—say, just less than the market price—and produces the amount which would give him the maximum profit at that price. For this "cut-throat" competition we have therefore the following postulate:

(*b*) *Each competitor regards the price as fixed and tries to make his profit a maximum.*

In this case we have the equation

$$\left(\frac{\partial \pi_1}{\partial u_1}\right)_{p \text{ constant}} = 0, \tag{10}$$

and, on account of the symmetry of the situation,

$$\left(\frac{\partial \pi_2}{\partial u_2}\right)_{p \text{ constant}} = 0$$

whence

$$0 = p - q'(u_1) = p - q'(u_2),$$
$$p = 2Au_1 + B = 2Au_2 + B.$$

[1] Cournot, *l. c.*, chapter V.

With reference to (4) then we have

$$p = \frac{Ab + B}{1 - Aa}, \qquad u_1 = u_2 = \frac{b + Ba}{2 - 2Aa}. \tag{11}$$

2. Phenomena with n producers. Let us denote the prices given by the postulates (a), (c), (m) and (b) respectively by letters p with a corresponding index. It is instructve to write down the formulæ for $p^{(a)}$, $p^{(c)}$ and $p^{(b)}$ when there are n producers instead of two. With reference to the postulates given in these cases, it is easily calculated that their values are the following:

$$p^{(a)} = \frac{b - 2Aab - nBa}{-a(n + 1 - 2Aa)} \quad \text{with} \quad u_i^{(a)} = \frac{b + Ba}{n + 1 - 2Aa}, \tag{12}$$

$$p^{(c)} = \frac{nb - 2Aab - nBa}{-a(2n - 2Aa)} \quad \text{with} \quad u_i^{(c)} = \frac{b + Ba}{2n - 2Aa}, \tag{13}$$

$$p^{(b)} = \frac{2Ab + nB}{n - 2Aa} \quad \text{with} \quad u_i^{(b)} = \frac{b + Ba}{n - 2Aa}, \tag{14}$$

and if we let n become large, we have the approximate formulæ

$$p^{(a)} = p^{(b)} = B \qquad \text{with} \qquad u_i^{(a)} = u_i^{(b)} = \frac{b + Ba}{n}, \tag{15}$$

$$p^{(c)} = \frac{1}{2}\left(B - \frac{b}{a}\right) \qquad \text{with} \qquad u_i^{(c)} = \frac{b + Ba}{2n}, \tag{16}$$

so that the total production in the case of coöperation would be about half of what it would be in either case of competition.

3. Nature of coefficients. In regard to the nature of the quantities which appear in all these formulæ, and which are the coefficients in the functions representing q and y, it may be pointed out that in the obviously typical case we have

$$a < 0, \qquad b > 0.$$

If, as has been done here, for $q(x)$ we regard the quadratic expression as an approximate representation of the cost function for all values of u, we may regard the A, B, C, determined statistically for this purpose, as all positive. In fact, C represents the cost of producing nothing per unit time, that is, the overhead expense, B represents the additional cost of producing the first unit, and the fact that A is positive represents the assumption that on the whole the cost curve is concave upward.

Since B is the cost of producing the first unit, and since both A and C are >0, the average cost per unit of any quantity u, which is q/u, must be $\geqq B$. Hence no industry will be started unless the demand y is positive for $p = B$; and therefore we assume, with reference to (2), $b + Ba > 0$. In this connection, notice equations (11) to (16), and (21) and (23′) below.

These hypotheses would not be legitimate if the quadratic expression were regarded as an approximation to the cost curve over an interval somewhat short. Thus, if the cost curve, more accurately represented, happened to be concave

downward in its first part, and concave upward in the second part, we should be able for the latter portion, which is the interesting region for the variable, to assume a quadratic expression in which A would be positive but not necessarily B and C. These possibilities are interesting when we wish to get some idea of the nature of taxation and its incidence, a subject in which generally our interest is confined to a small portion of the cost curve. But for a rough approximation to the phenomena under investigation we may return to our original assumptions, with $A, B, C > 0$. It is justifiable also to assume all our producers working with the same cost curve, since we wish to obtain not an idea of the effect of a divergence of cost curves, but a rough notion of the phenomenon of competition itself.

There are one or two special cases which we may note in passing. If we put $n = 1$, the prices $p^{(a)}$, $p^{(c)}$ become identical with the $p^{(m)}$ for monopoly. This is not the case however with $p^{(b)}$ which reduces to

$$p^{(b)} = \frac{2Ab + B}{1 - 2Aa}, \qquad \text{with} \qquad u^{(b)} = \frac{b + Ba}{1 - 2Aa}. \tag{17}$$

In fact, since $-2Aa > 0$, the u for (17) is greater than the u for monopoly given by (9), and accordingly $p^{(b)} < p^{(m)}$.

Another situation to note is where the demand y is independent of the price, *i.e.*, where $a = 0$. In this case

$$p^{(m)} = p^{(c)} = p^{(a)} = \infty, \tag{18}$$

or in other words, the prices are pushed up beyond the region in which the hypothesis of demand independent of price remains valid, not only for monopoly and coöperation, but also for competition as described by Cournot. On the other hand, for $p^{(b)}$ we get the value

$$p^{(b)} = 2A\frac{b}{n} + B, \qquad \text{with} \qquad u_i^{(b)} = \frac{b}{n}. \tag{18'}$$

The two kinds of competition represent therefore quite distinct situations.

Some essential differences are brought out if we consider an industry where aside from overhead cost most of the cost is labor—a situation which we may characterize by writing $A = 0$. We obtain

$$p^{(a)} = \frac{n}{n+1}B - \frac{b}{(n+1)a},$$
$$p^{(c)} = p^{(m)} = \frac{1}{2}B - \frac{b}{2a},$$
$$p^{(b)} = B.$$

4. Relative values. There are simple inequalities that hold in general for the various kinds of prices. In fact, from (12), (13), (14) it follows obviously that

$$u_i^{(c)} < u_i^{(a)} < u_i^{(b)},$$

for any particular value of n, $n > 1$. Hence, since $y = \Sigma u_i = nu_i$,

$$y^{(c)} < y^{(a)} < y^{(b)},$$

and since $y = ap + b$,

$$p^{(c)} > p^{(a)} > p^{(b)}. \tag{19}$$

Moreover from (13),

$$y^{(c)} = nu_i^{(c)} = \frac{b + Ba}{2 - \dfrac{2Aa}{n}}.$$

But the quantity $-Aa$ being positive,

$$\frac{b + Ba}{2 - \dfrac{2Aa}{n}} > \frac{b + Ba}{2 - 2Aa},$$

that is,

$$y^{(c)} > y^{(m)},$$

and therefore,

$$p^{(c)} < p^{(m)}. \tag{19$'$}$$

Hence, finally, for any value of n, $n > 1$, the prices are arranged in descending order, as follows: monopoly, coöperation, competition (a), and competition (b). The total productions are arranged in the reverse order. It is possible, but hardly worth while for the present paper, to get simple formulæ for the direct measure of these differences.

5. Failure. If the price falls below a certain value, any producer will fail to make a profit; and if it falls still lower, since $C \neq 0$, the producer will lose at a definite rate. It remains to find this critical price.

The line $q = pu_i$ cuts the curve $q = Au_i^2 + Bu_i + C$ in two points, real, coincident or imaginary. In the first case only is it possible to obtain values of u_i for which $pu_i > q(u_i)$, *i.e.*, for which there shall be a profit. The two points are determined by the condition

$$0 = Au_i^2 + (B - p)u_i + C,$$

and will therefore be coincident when

$$(B - p)^2 = 4AC,$$

that is, when

$$p = B + 2\sqrt{AC},$$

the plus sign being taken with the radical, since the other value of p would correspond to a negative value of u. This is deduced immediately from the shape of the curve for q.

The condition $p^{(b)} \geqq p$ is equivalent to the condition

$$nB + 2Ab \geqq (n - 2Aa)(B + 2\sqrt{AC}),$$

from which a necessary condition for profit, under postulate (b), follows:

$$n < \sqrt{\frac{A}{C}}(b + Ba + 2a\sqrt{AC}). \tag{21}$$

This, it turns out, is under (*b*) also a sufficient condition, for the value of u_i is determined by the equation $p = q'(u_i)$, and the value of u_i which corresponds to p by this equation must by the law of the mean lie between the two intersections which the line $q = pu_i$ makes with the graph of q, when these are real. We have here then a definite finite limit on the number of competitors who may safely engage in a given production. Similar limitations apply to the cases governed by the other postulates (*a*) and (*c*); in all such limitations the importance of overhead expense is manifest, as in (21).

6. Offer. Equation (17) represents a sort of monopoly price. If there is a single producer, and if for each price he would produce the amount which would give him the maximum profit at that price, then the price in the market and the amount of goods produced and sold in unit time would be given by (17). In this case then (which hardly corresponds to any true monopoly) the producer regards himself as having a definite offer of the commodity for each price p, and the price in the market is obtained by observing the intersection of the curves of offer and demand. In a similar fashion, in the case (*b*) in general, each producer regards his individual offer as a function of the price, and the total offer is again defined. Offer is the amount of the commodity which would, if the price were p, be placed on the market in unit time.

It is most important to notice that in the general cases (*a*), (*c*), (*m*) offer in this sense has no relation to the problem, and the offer-demand diagrams lead to wrong results. In fact, in the problems (*m*), (*c*) and even (*a*) (unless n is so large that the quantity u_i/a in (5′) may be neglected, and (*a*) and (*b*) become thereby identical), the quantity u_i cannot be obtained as a function of p without introducing information (in $\partial p/\partial u_i$) derived from the demand curve. Thus offer as ordinarily interpreted loses its significance. On the other hand, any other definition of offer would be artificial, and the calculation of a new curve to fit into the offer-demand diagram, in order to give the proper intersection, would involve knowing in advance the solution of the problem.

6.1. One sees in this way that there may not be much sacredness in the "law of supply and demand," as an objection to the fixing of prices. The effect of price fixing is to make the situation entirely a type (*b*) problem, and thus to create an offer of goods in the technical sense, as we have just defined it, and the offer so created may be actually greater than the amount of production would be, which would give the maximum profit under less restricted conditions. It is in this way that the equations (17) become of practical interest, as portraying the case (*b*) when there is only one producer.

As an instance, suppose, let us say, that under war-time conditions the quantities q and y are given by functions (1) and (2), with certain values for the coefficients, and it is required to find what price $p_0 < p^{(m)}$ may be imposed on a monopoly product, without changing the amount produced. If the price is fixed at p_0, the relation of u to p_0 is fixed by postulate (*b*), so that u is given by the equation

$$p_0 = 2Au + B, \qquad \text{whence} \qquad u = \frac{p_0 - B}{2A}\,. \tag{22}$$

From this equation and (9) we have

$$\frac{p_0 - B}{2A} = \frac{b + Ba}{2 - 2Aa},$$

so that, finally,

$$p_0 = \frac{B + Ab}{1 - Aa}. \tag{23}$$

If the price were not fixed, it would become $p^{(m)}$, given by (9), and the relation between the two prices is therefore the following:

$$p^{(m)} - p_0 = \frac{b + Ba}{-a(2 - 2Aa)}, \tag{23'}$$

in which, in fact, the right-hand member denotes a positive quantity.

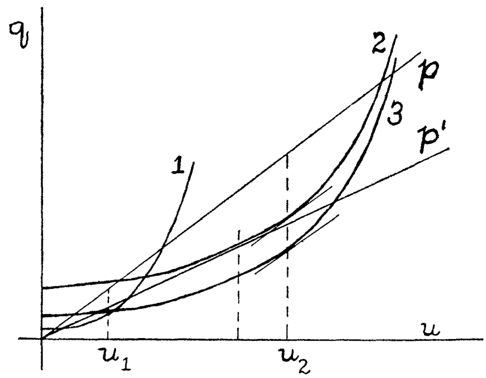
6.2. Before completing this subject, let us consider very briefly a related one. Is it possible to tax an industry in such a way that prices will not be changed? We notice in fact that a change in C has no effect on the price, so that either a tax in the shape of a fine of so many dollars a year, or a bounty of so many dollars a year will leave the price and the amount produced unchanged. In general, if we denote by $\bar{\pi}(u)$ the profit under the modified conditions, it is sufficient for the price not to change that we have

$$\bar{\pi}_i(u_i) = \alpha\pi_i(u_i) + \beta_i,$$

where α and β_i are constants, for in this case $\bar{\pi}_i'(u_i)$ and $\pi_i'(u_i)$ vanish for the same value of u_i. In other words, any procedure which takes away a definite proportion of the profit, or a definite amount, per unit time, or both, will not modify the price. We must guard against saying that an income tax will not modify prices, for an income tax affects a considerable proportion of incomes, and will thus presumably affect the a and b of the demand function for practically all commodities. An excess profit tax, however, is not open to this qualification, and satisfies the above conditions.

One may fix the price against a monopoly by (23), and then fine it by excess profit taxes, charter taxes, confiscation of dividends, and so on, however unjust the process may be, without diminishing the rate of production in the slightest degree. And thus, in connection with (23), it may be that the price p_0 will cause a loss rather than a profit, and this can only be met by passing dividends, or even going into a receiver's hands; yet the production will not diminish. On the other hand, if the price is fixed at less than p_0, the production will diminish even if there is still a profit, and even if the cost curve is modified by passing dividends and failing to pay interest on bonds. For these are modifications of the C in the expression for $q(u)$ and have no effect on the maxima and minima involved. In exceptional times it may be quite desirable by fixing prices at restrictive values to decrease the production in certain industries, say luxuries, and increase competition in basic industries and new necessities, with the object of keeping the price system as a whole fairly stable.

7. Modifications in the cost curve. Rather than consider the total variety of cost curves among producers, let us consider a type of modification which is characteristic in our problem. When prices are in a certain condition, under postulate (b), with say n competitors producing at a profit, some of the producers may wish to make a larger profit by increasing their overhead expenses, decreasing their other expenses, and producing larger amounts. In the diagram, the curves (1) and (2) represent an individual cost curve in these two states, the amount which would be produced in each case being that which would make the distance between the line $q = pu$ (which represents the selling value) and the cost curve a maximum. This is the position in which the tangent to the cost curve is parallel to the price line $q = pu$. The transition contemplated is from (1) to (2).



In (b), as we have seen, an offer curve is defined, so that offer is given as a function of price. The transition from (1) to (2), however, changes this function, and as is evident from the remark just made, will increase the offer at the price p. Since $a < 0$, this will decrease the price, and if the price goes below a certain amount p', indicated on the diagram by the line $q = p'u$, there will no longer be any profit in the expanded business. The change from (1) to (2) is unfortunately usually not a reversible process. And the usual modification of the cost curve which is now enforced is from (2) to (3), by omitting to pay dividends, and by such other changes in the value of C as ultimately provoke the bondholders to demand a receivership. Thus even apart from any phenomenon connected with the rate of interest,[1] bankruptcy may be regarded as a normal event in a system of competition.

8. General points of view. In a general system of economics we cannot for a proper discussion restrict ourselves to functions of a single variable. Mathematically this is not so essential a modification as it has been sometimes regarded. An extension of the problem which goes deeper is what is obtained when we remember that what a producer is interested in is not to make his momentary profit a maximum, but his total profit over a period of time of considerable extent, with reference to cost functions which are themselves changing as a whole with respect to time, such as in the instance discussed in the previous section. The mathematical discipline which enables us to find functions which make a maximum or a minimum quantities which depend upon them throughout periods of time is the calculus of functionals, or in special cases the calculus of variations. But the quantity which we want to make a maximum over a period of time

[1] Irving Fisher, *The Purchasing Power of Money*, New York, 1913; see chapter IV.

need not be the total profit; it may be the total production, or whatever other quantity we wish to take as a desirable characteristic of the social system we discuss. The author regrets that at the present time he can refer only to his lecture courses for a further treatment of this point of view. Nevertheless it seems the most fruitful way that a really theoretical economics may be developed.

ON KELLOGG'S DIOPHANTINE PROBLEM.[1]

By D. R. CURTISS, Northwestern University.

1. Kellogg's Problem. Two Applications. In a recent number of the MONTHLY,[2] Professor Kellogg has presented a very interesting discussion of the Diophantine equation

$$\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}=1, \tag{1}$$

in which he gives reasons for believing that the maximum value of any of the unknowns that can occur in a solution in positive integers is u_n, where

$$u_1 = 1, \qquad u_{k+1} = u_k(u_k + 1). \tag{2}$$

Thus the successive u's are 1, 2, 6, 42, 1806, $\cdots$. I propose here to give a proof of the correctness of this statement, a proof in which we use sequences of inequalities, each containing one less x than the preceding. The method may be of some interest in itself, and of some value in similar problems. That the proof is hardly so simple as the statement of the problem will cause no surprise, at least to one familiar with Diophantine analysis.

Before we take up this proof it may add some interest to note two problems, one geometrical, the other arithmetical, whose solution depends on finding particular sets of integers satisfying (1). The first is that of laying non-overlapping sets of floor-tiles, each tile being a regular polygon whose sides are of unit length, so as to cover the plane, or a portion of the plane, just once or multiply; the polygons of a set will not, in general, all have the same number of sides. For example, suppose we are to fit n such tiles against each other so that all shall have a common vertex, and the piece of surface formed by them shall wind k times about this vertex, the last tile being in such a position as to fit without overlapping against the first tile if it were in the first layer instead of the last. In other words, the tiles are to generate without overlapping a piece of a Riemann surface in which the sheets form one cycle about a branch point. If the n tiles

[1] Read before the American Mathematical Society, December, 1921.

[2] *1921*, 300–303. See references there to Carmichael's *Diophantine Analysis* and to his review of Dickson's *History of the Theory of Numbers*, vol. 2, in this MONTHLY. This subject is very close to that of Sylvester's paper, "On a point in the theory of vulgar fractions," *American Journal of Mathematics*, vol. 3, 1880, pp. 332–335 and 388–389, which, Sylvester says, was suggested by the account in Cantor's *Geschichte der Mathematik* of the ancient Egyptian treatment of fractions by resolution into a sum of fractions each having unity as its numerator.